

Spectral Flow and Quantum Theory of Dissipation in the Vortex Core of BCS superconductors

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The dissipation process in two dimensional BCS superconductor due to the quasiparticles in the core of moving vortex is studied from a quantum mechanical point of view, especially paying attention to the spectral flow of the bound state energies in the vortex core. In order to clarify the nature of the spectral flow, we performed a numerical study of a finite system and, by extending the analysis to include the effect of impurities, we discuss the quantum mechanical origin of the dissipation in the vortex core.

KEYWORDS: superconductivity, vortex, spectral flow, dissipation

§1. Introduction

Recently the vortices in type-II superconductors in the quantum limit, where the level spacing of the Caroli-de Gennes-Matricon's (CdGM's) bound states in the vortex core, $\hbar\omega_0 = \Delta_0^2/E_F$ (Δ_0 and E_F are bulk energy gap and Fermi energy, respectively),¹⁾ is larger than the energy scale of thermal fluctuation, $k_B T$ (k_B and T are Boltzmann constant and temperature, respectively), have been attracting much attention stimulated by the study of high T_c oxide and organic superconductors. Especially in the moving vortices, the effects of bound states are believed to be essential in understanding the vortex dynamics. In order to treat this problem, an interesting idea has been proposed, that is the spectral flow.^{2,3,4)} This is based on the finding that the bound state energies in the vortex core change continuously when the vortex is moving. Employing this idea, the nondissipative part of the vortex dynamics has been studied intensively, although there still remains controversy.^{6,5)}

On the other hands, it is also an interesting problem to see how the dissipative part of vortex dynamics is modified from the Bardeen-Stephen's classical theory⁷⁾ in the quantum limit. Several attempts have been made to clarify this point. Until now, however, most of the studies were focused on the transition between CdGM's bound states due to the interaction with impurities.^{8,9,10,11,12)} In this letter, we argue that the transition between CdGM's bound states is driven by the spectral flow although it is significantly modified by the impurities. This point of view provides us with some new knowledge about the dynamics of quasiparticles in the core of moving vortices.

In this letter, in order to clarify the nature of the spectral flow, we first perform a finite size study of the energy levels of quasiparticles in the presence of vortices. We consider a ring made from a strip of two dimensional

(2D) BCS superconductor and concentrate on the case of single vortex in the system. The numerical evaluation of the eigenvalues of Bogoliubov-de Gennes equation clearly shows a spectral flow behavior. It is also shown that there are two kinds of spectral flow; one moving from the upper side to the lower side of the gap and the other moving in the opposite direction, which we call ε_+ - and ε_- -branch, respectively.

Based on these facts, we argue that the transition of an electron from ε_- to ε_+ -branch, which occurs due to impurities, gives the microscopic description of the quasiparticle scattering, *i.e.* the dissipation in the vortex core. This process is studied from a quantum mechanical point of view employing Landau-Zener tunneling theory and it is clarified when the Bardeen-Stephen type description is valid.

This letter is organized as follows: In Sec. 2 we describe our model and basic scheme for the numerical calculation. Then the numerical result is introduced, based on which the nature of the spectral flow is discussed. In Sec. 3 the effect of impurities is considered and the mechanism of the dissipation in the vortex core is discussed. In Sec. 4 we discuss the relation between our results and the preceding studies. Some further problems are also mentioned. In Sec. 5 we summarize our results.

§2. Bogoliubov-de Gennes equation

In this letter, we consider a ring made from a strip of 2D BCS superconductor (Fig. 1), whose wave functions and energy spectrum of the quasiparticles are given by the Bogoliubov-de Gennes equation. The width and length of the strip are given by w and l , respectively. Magnetic flux Φ is trapped in the cylinder to generate circulating supercurrent. We consider the case of single vortex located at $\mathbf{R}(t) = (R_x, R_y)$ at time t .

Bogoliubov-de Gennes equation is given by,¹³⁾

$$\begin{pmatrix} \xi(\mathbf{r}) & \Delta(\mathbf{r}, t) \\ \Delta^*(\mathbf{r}, t) & -\xi^*(\mathbf{r}) \end{pmatrix} \cdot \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \epsilon_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \quad (2.1)$$

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where

$$\xi = -\frac{\hbar^2}{2m} \left(\nabla - \frac{ie}{\hbar c} \mathbf{A}_0 \right)^2 - E_F + V_{\text{imp}}(\mathbf{r}) \quad (2.2)$$

with m , e and c being the mass and charge ($e < 0$) of electrons, and the velocity of light, respectively; $\mathbf{A}_0 = (\Phi/l) \hat{\mathbf{e}}_x$ with $\hat{\mathbf{e}}_x$ being unit vector in x -direction; u_n and v_n are the wave functions of the quasiparticles and ϵ_n is their energy eigenvalue; $V_{\text{imp}}(\mathbf{r})$ is the impurity potential. We disregard the fluctuation of the gauge field in this letter.

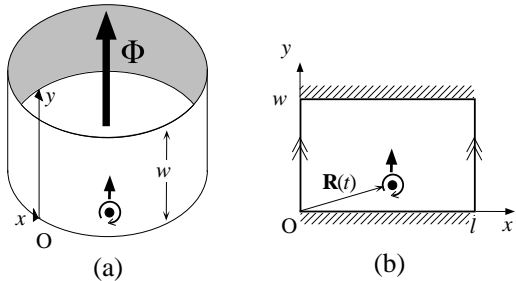


Fig. 1. (a) Picture of the model system which we consider in this letter. Magnetic flux Φ is introduced to generate circulating supercurrent in the ring. A vortex (indicated by a dot) crosses the ring and causes decay of the supercurrent. (b) Equivalent area in x - y plane. Lines at $x = 0$ and $x = l$ are identified. The position of the vortex is denoted by $\mathbf{R}(t)$.

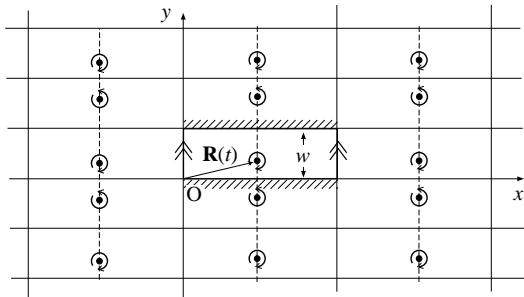


Fig. 2. Infinite number of mirror vortices generated by the boundary conditions of the system (in the middle).

First we solve Eq. (2.1) with $V_{\text{imp}}(\mathbf{r}) = 0$ by introducing an approximate solution for $\Delta(\mathbf{r}, t)$, instead of solving it selfconsistently. The approximate solution is constructed in the following way. It is well known that the condition of no outgoing supercurrent at the boundary is satisfied by introducing mirror vortices. In the present case an infinite series of mirror vortices appears, as is seen from Fig. 2, which is considered to be two square vortex lattices with opposite flux overlapped with each other with a shift in y direction. Therefore we can employ the order parameter of vortex lattice to construct an approximate order parameter for the present system. We use Eilenberger's order parameter¹⁴⁾ for the vortex lattice and apply a correction to the amplitude part in the sense of Clem's approximation,¹⁵⁾ in order to make

the amplitude constant except for the core region;

$$\begin{aligned} \phi_{\pm}(\mathbf{r}) &= e^{-\pi(\bar{y} \mp \bar{R}_y + \bar{w})^2} \vartheta_3(\bar{x} + i(\bar{y} \mp \bar{R}_y + \bar{w}), 2\bar{w}i) \\ \tilde{\phi}_{\pm} &= \phi_{\pm} / \sqrt{|\phi_{\pm}|^2 + c} \\ \Delta &= \Delta_0 \tilde{\phi}_+ \tilde{\phi}_-^*, \end{aligned} \quad (2.3)$$

where $\vartheta_3(z, \tau)$ is the elliptic theta function; 'bar's denote lengths divided by l , e.g., $\bar{x} = x/l$; c is a positive constant which determines the radius of the core; Δ_0 is the equilibrium energy gap outside the vortex core; $\tilde{\phi}_-^*$ is the complex conjugate of $\tilde{\phi}_-$.

We assumed parabolic dispersion for the Fermions and took into account only the states whose energy E satisfies $E_F - \hbar\omega_D < E < E_F + \hbar\omega_D$ where $\hbar\omega_D$ is a cut-off parameter. The matrix elements of the Hamiltonian (i.e., the matrix of the left hand side of Eq. (2.1)) were calculated using this basis and energy eigenvalues were obtained after diagonalization.

In Fig. 3, we show the result for the case of $w = 0.5 \times l$, $E_F = 10^3 \times e_0$, $\Delta_0 = 10^2 \times e_0$ and $\hbar\omega_D = 300 \times e_0$ where $e_0 = (\hbar^2/2m)(2\pi/l)^2$. Here $\xi = \hbar v_F/\Delta_0 = \pi^{-1}l(E_F/\Delta_0)\sqrt{e_0/E_F}$. Although the system is much smaller than the realistic one, our result shows a clear spectral flow behavior.

Here we note a remarkable character of the spectral flow, namely the existence of many level crossings. This is the most essential process which leads to the dissipation in the vortex core, as we see in the next section. This character is understood in the following way: If (u_n, v_n) is an eigenfunction with eigenvalue ϵ_n , (v_n^*, u_n^*) is also an eigenfunction with eigenvalue $-\epsilon_n$.¹³⁾ Therefore the energy spectrum is symmetric with respect to the Fermi energy. This means that if there are several spectral flow branches which are flowing down crossing the Fermi energy, the same number of branches must be flowing up. These branches inevitably make crossing points.

§3. Effect of impurity scattering

In this section we study the effect of impurities $V_{\text{imp}}(\mathbf{r})$. For this purpose it is useful here to look at the dynamics of quasiparticles in superconductor-normal metal-superconductor (S - N - S) Josephson junction,^{16,17)} which shows a similar spectral flow behavior.⁴⁾ In S - N - S Josephson junctions, the bound states are formed in N -region due to Andreev scattering at S - N boundaries and, if we change the phase difference of two S 's, the energy levels of the bound states also change continuously.¹⁶⁾ We can also see level crossings similar with our system. When a finite bias voltage is applied to the junction, the phase difference increases in time as is known from the Josephson's acceleration relation and the spectral flow occurs.

A precise analysis of this system, when N region is a point contact, was given by Averin and Bardas.¹⁷⁾ They introduced a potential barrier in N -region and found that the scattering of quasiparticles by the barrier causes splitting of the crossing points of the spectral flow. The dominant scattering process is the backward scattering and the quasiparticles are scattered into the states with

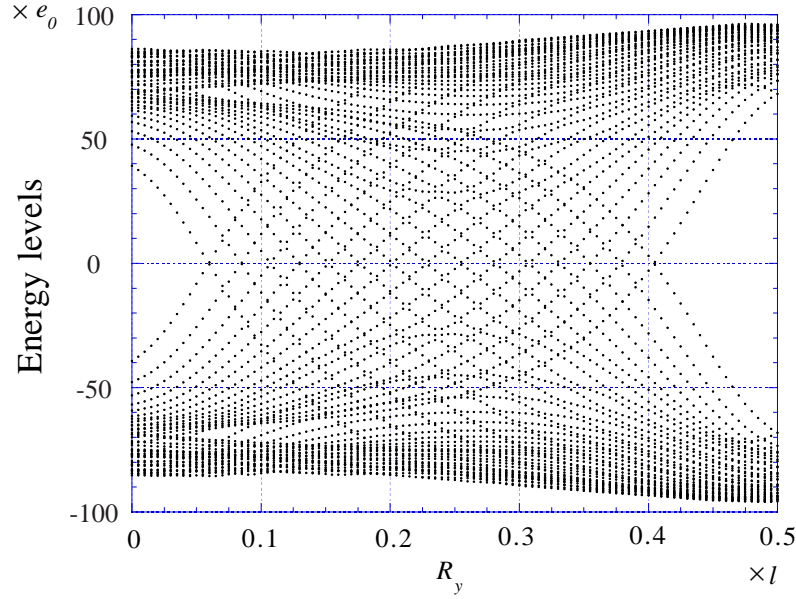


Fig. 3. Energy spectrum of quasiparticles as a function of R_y . We can see the energy gap opening at $R_y = 0$ and $R_y = w = 0.5 \times l$, and many energy level crossings in the middle. The crossing points are almost uniformly distributed in the range $0 < R_y < w$ except for the region which is approximately within the coherence length $\xi \sim 0.1 \times l$ from the boundaries. The gap at $R_y = w$ is smaller than $\Delta_0 = 10^2 \times e_0$, because the average of the gap is suppressed by the effect of finite system size. The gap at $R_y = 0$ is suppressed further, because the supercurrent is larger at $R_y = 0$ than at $R_y = w$. Some of the energy levels outside the gap are not plotted. At some points, splitting of the crossing points is seen, which is due to the approximation and not essential.

opposite momentum. They also found that the dynamics of the scattering is describable employing Landau-Zener tunneling theory.

From the analogy to the Josephson junctions, we can probably expect that the similar process occurs due to impurity scattering at each crossing points of Fig. 3. We can approximately estimate the magnitude of the splitting as \hbar/τ_{imp} where τ_{imp} is the life time of quasiparticles in the normal state due to impurity scattering. When the vortex is moving with a velocity v_R , the scattering process can be described by the Landau-Zener tunneling theory. The time evolution of the system is shown in Fig. 4 (a): If both of two crossing levels are vacant (or occupied) before crossing, the problem becomes rather trivial, since nothing special happens at the crossing point. However, if the lower level is occupied and the upper one is not, the final state changes depending on v_R . When v_R is small, the particle follows the adiabatic change of the energy level, *i.e.*, the broken curve in Fig. 4 (a). When v_R is large, the particle has little time to interact with impurities and travels along the original levels without being scattered, *i.e.*, dotted line. The probability of the latter case is given by

$$p = \exp \left\{ -\frac{2\pi(\hbar/\tau_{\text{imp}})^2}{\hbar(\dot{\epsilon}_+ - \dot{\epsilon}_-)} \right\} \quad (3.1)$$

where $\dot{\epsilon}_+ - \dot{\epsilon}_-$ is the relative velocity of two levels in energy space, which is estimated as follows: At low temperatures $k_B T < \hbar\omega_0$, only the level crossings near the Fermi energy are relevant, which is depicted in Fig. 4 (b). Firstly, the maximum of the spacing between two levels nearest to the Fermi energy is of the order of $\hbar\omega_0$,

which approximately correspond to the CdGM's bound states with angular momentum $\mu = 1/2$ and $-1/2$. Secondly, in our system the wave number of electrons in y -direction is quantized by π/w . Therefore there exist $k_F/(\pi/w)$ channels, each of which is responsible for one pair of spectral flow (ϵ_+ and ϵ_-). If we assume that the crossing points near the Fermi energy are distributed uniformly in $0 < y < w$ when $\xi \ll w$, we can estimate the separation between crossing points to be π/k_F , a half of the Fermi wave length. Although this is just an assumption at this stage, it seems to be a natural one, as one can see from Fig. 3. Therefore, noting $R_y = v_R t$, we obtain $\dot{\epsilon}_+ - \dot{\epsilon}_- \simeq \hbar\omega_0 k_F v_R / \pi$.

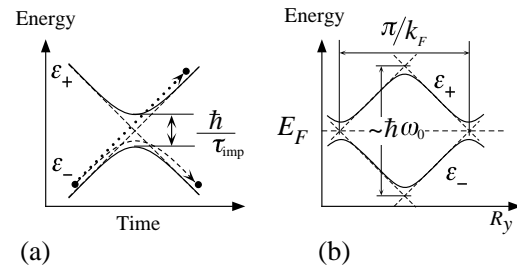


Fig. 4. (a) Level crossing and Landau-Zener tunneling process. (b) The energy and length scale near the crossing point.

We can see from Eq. (3.1) that when $v_R k_F \tau_{\text{imp}}^2 \omega_0 \ll 1$, the quasiparticle follows the change of the energy level adiabatically, which means that the quasiparticle is scattered by the impurity. In contrast with this, when

$v_R k_F \tau_{\text{imp}}^2 \omega_0 \gg 1$, the quasiparticle is not scattered but becomes a particle-hole pair after some level crossings, which costs approximately the energy of Δ_0 . This energy cost may probably suppress the vortex motion by acting as a nondissipative force. Further studies are needed to clarify the detail of this process, however.

It is also important to note that two different situations arise from the relation between the level spacing $\hbar\omega_0$ and the magnitude of splitting \hbar/τ_{imp} . When $\hbar\omega_0 \gg \hbar/\tau_{\text{imp}}$ the situation is depicted in Fig. 5 (a). The energy levels oscillates with a period of a half of the Fermi wave length. In this case, a periodic potential for the vortex may appear due to the change of the quasiparticle energy. If we simply sum up the energies of the quasiparticles in the bound states, the amplitude of the periodic potential becomes the order of $\hbar\omega_0$, which is not negligible at lower temperatures $k_B T < \hbar\omega_0$. On the contrary, when $\hbar\omega_0 \ll \hbar/\tau_{\text{imp}}$, the system recovers translational symmetry, as is seen from Fig. 4 (b).

The dissipation of Bardeen-Stephen type is, therefore, observed when $v_R k_F \tau_{\text{imp}}^2 \omega_0 \ll 1$ and $\omega_0 \ll 1/\tau_{\text{imp}}$ are satisfied at the same time.

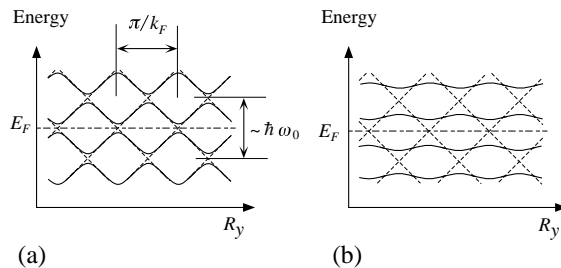


Fig. 5. Schematic view of the energy levels of the bound states in the case of (a) $\omega_0 \gg 1/\tau_{\text{imp}}$ and (b) $\omega_0 \ll 1/\tau_{\text{imp}}$.

§4. Discussion

In the pioneering study, Caroli, de Gennes and Matricon assumed that the vortex has rotational symmetry.¹⁾ Therefore the angular momentum was taken to be a good quantum number. This, however, is not the case with our system, in which the boundaries break the rotational symmetry and cause mixing of different angular momentum states. This is the origin of the spectral flow in this letter. Volovik, Makhlin and Misirpashaev,^{2,3)} by introducing the *laboratory frame*, obtained a similar spectral flow, which we consider is a realization of our finite size calculations in the bulk. The relation between two, however, is not clear at this stage.

There are several works which have studied the dissipation in the vortex core from a quantum mechanical point of view, by addressing the transition of quasiparticles between different CdGM's bound states due to impurity scattering.^{10,9,11,12)} The largest difference of our study from those is that the transition of quasiparticles is driven by the spectral flow which exists even without impurity scattering. In our treatment, however, the effect of impurity scattering is only introduced by τ_{imp} which is too simplified, especially in the case of small

concentration of impurities. If we treat this more exactly, mesoscopic fluctuations may appear in the energy levels of Fig. 5, reflecting the character of individual impurities.^{11,12)}

In this letter we neglected the gauge fluctuation, namely the effect of finite penetration depth. Since the mirror vortices and their long-ranged phase modulation are important in our analysis, the effect of gauge fluctuation can be crucial. In order to clarify this point, we have to include the effect of gauge field into our Bogoliubov-de Gennes equation, which however needs further studies.

In actual superconductors, three dimensionality is also important. In this case, the vortex become a line and each bound state in the vortex core obtains a continuous level corresponding to the motion along the line, which, of course, makes the problem much more complicated. However, since the dissipation is mostly due to the backward scattering of quasiparticles, which is expected to occur between the states with the same momentum in the direction of the line, we may believe that the physical picture presented in this letter is still useful to understand the basic mechanism of the dissipation.

§5. Summary

In this letter we studied the spectral flow and the origin of dissipation in the core of superconducting vortices from a quantum mechanical point of view. We found that several characteristic regimes are defined by the material parameters and the vortex velocity. The dissipation of Bardeen-Stephen type occurs only when $v_R k_F \tau_{\text{imp}}^2 \omega_0 \ll 1$ and $\omega_0 \ll 1/\tau_{\text{imp}}$ are satisfied. In other regimes the effect of the spectral flow does not result in simple dissipation but can cause nondissipative forces as well.

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